EPFL

Efficient Interpolation between Extragradient and Proximal Methods for Weak MVIs

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Motivation

Minimax

nonsmooth (i.e. indicator of a constraint set)

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \underset{y\in\mathbb{R}^m}{\text{maximize}}\ g(x) + \varphi(x,y) - h(y)$$

Smooth

- **Minimization**: local solutions can be found efficiently for nonconvex
- **Minimax**: even a local solution is in general intractable
  - [Hirsch & Vavasis, 1987, Papadimitriou, 1994, Daskalakis et al., 2021]
  - exemplified by the extragradient method converging to a limit cycle (Fig. 1)

In todays talk:

- Fundamental question of what nonmonotonicity **first order methods** can handle
- **Separation** between extragradient and prox-based methods (in contrast with monotone)
- We will shave off a **logarithmic factor** in the literature

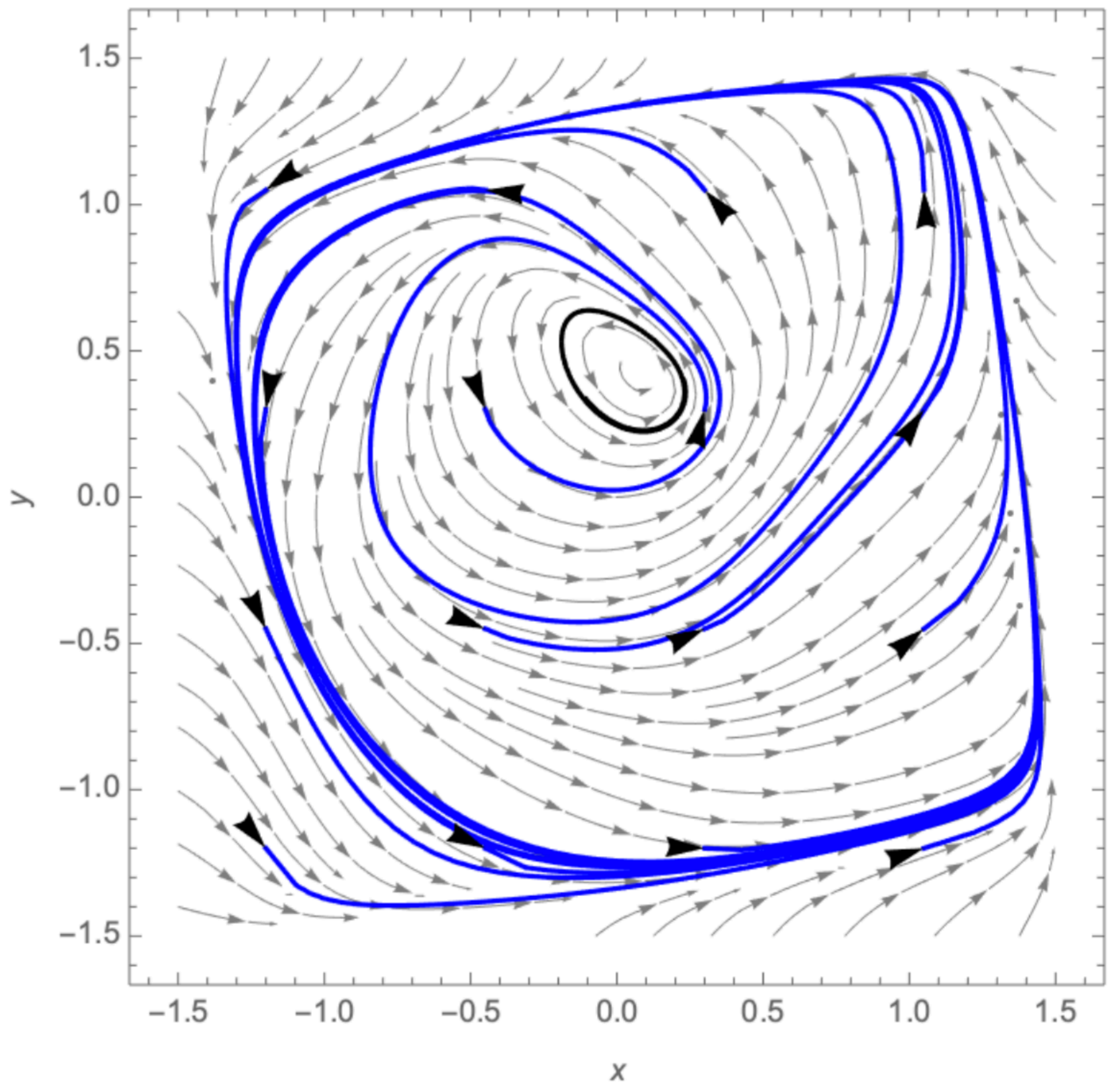


Figure 1. [Hsieh et al., 2021, Ex. 5.2]

Problem formulation

$$\underset{x\in\mathbb{R}^n}{\text{minimize}}\ \underset{y\in\mathbb{R}^m}{\text{maximize}}\ g(x) + \varphi(x,y) - h(y)$$

Find $z \in \mathbb{R}^d$ **such that**

Captures first order condition of

$$0 \in Fz + Az =: Sz$$

Operator view: $z = (x, y)$

$$Fz = (\nabla_x \varphi(x,y), -\nabla_y \varphi(x,y))$$

$$Az = (\partial g(x), \partial h(y))$$

$$(\mathrm{id} + \gamma A)^{-1}(z) = (\mathrm{prox}_{\gamma g}(x), \mathrm{prox}_{\gamma h}(x))$$

Assumption 1 $F : \mathbb{R}^d \to \mathbb{R}^d$ is L-Lipschitz

Assumption 2 $A : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ is maximally monotone

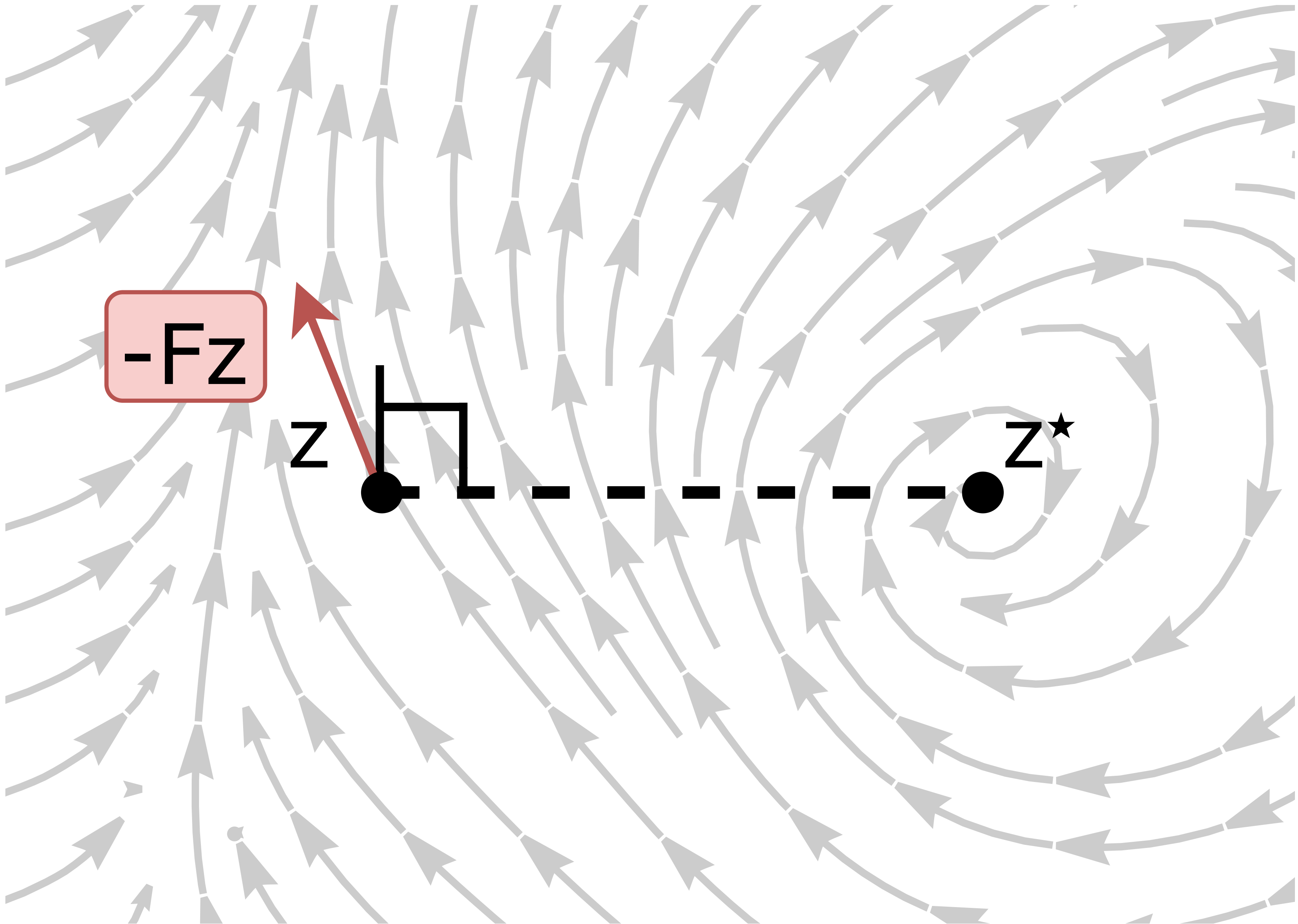
Assumption 3 $S : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ satisfies the weak Minty variational inequality (MVI), i.e. for all $z^\star \in \mathcal{Z}^\star \subseteq \operatorname{zer} S$ (nonempty $\mathcal{Z}^\star$) and some $\rho \in (-\frac{1}{L}, \infty)$

$$\boxed{\langle v, z - z^\star \rangle \geqslant \rho \|v\|^2} \quad \text{for all} \quad (v, z) \in \operatorname{grph} S$$

Nonmonotone when negative!

$$\Downarrow (A \equiv 0)$$

$$\langle Fz, z - z^\star \rangle \geqslant \rho \|Fz\|^2$$



Why do we care?

- The structure can capture [Hsieh et al.,2021,Ex. 5.2]
- Turns out to be fundamental (pops out of the analysis)

Existing literature

Assumption (Weak MVI)
$\langle v, z - z^\star \rangle \geqslant \rho \|v\|^2$ for all $(v, z) \in \operatorname{grph} S$

[P. et al., 2022] [Fan et al., 2023] [Alacaoglu et al., 2024]

0 $\quad \rho > -1/2L \quad \rho > -(1-1/e)/L \quad \rho > -1/L \quad \rho$

Monotone (w.r.t. solution)

Negative $\rho \Rightarrow$ increasingly nonmonotone

relaxed update: $z^{k+1} = (1-\alpha) z^k - \alpha \bar{z}^k$
$(A \equiv 0)$

$$\begin{cases} \bar{z}^k = z^k - \gamma F(z^k - \gamma F z^k) & \text{(extragradient)} \Rightarrow \rho > -1/2L \\ \bar{z}^k = z^k - \gamma (F \bar{z}^k - \varepsilon^k) & \text{(proximal point)} \Rightarrow \rho > -1/L \end{cases}$$

Method	Minimum ρ	Complexity[1]	Interpolates	Stopping criteria[2]	Constraints	Fejér monotone
Pethick et al. [2022]	$-\frac{1}{2L}$	$\mathcal{O}(\frac{1}{\varepsilon})$	✗	-	✓	✓
Fan et al. [2023]	$-\frac{1-1/e}{L}$	$\mathcal{O}(\frac{1}{\varepsilon})$	✓	✗	✗	✓
Alacaoglu et al. [2024]	$-\frac{1}{L}$	$\mathcal{O}(\frac{1}{\varepsilon} \ln \frac{1}{\varepsilon})$	✗	✗	✓	✗
This paper	$-\frac{1}{L}$	$\mathcal{O}(\frac{1}{\varepsilon})$	✓	✓	✓	✓

Main contribution $\rho > -1/L$ without suffering a logarithmic factor!

Key insight
- Inaccuracy in the halfspace projection can both correct for a proximal approximation and enlarge the problem class
- We will extend ideas from Solodov and Svaiter [1999] and P. et al. [2022]

A Hybrid method

Assumption (Weak MVI)

$\langle v, z - z^\star \rangle \geqslant \rho \|v\|^2$ for all $(v, z) \in \operatorname{grph} S$

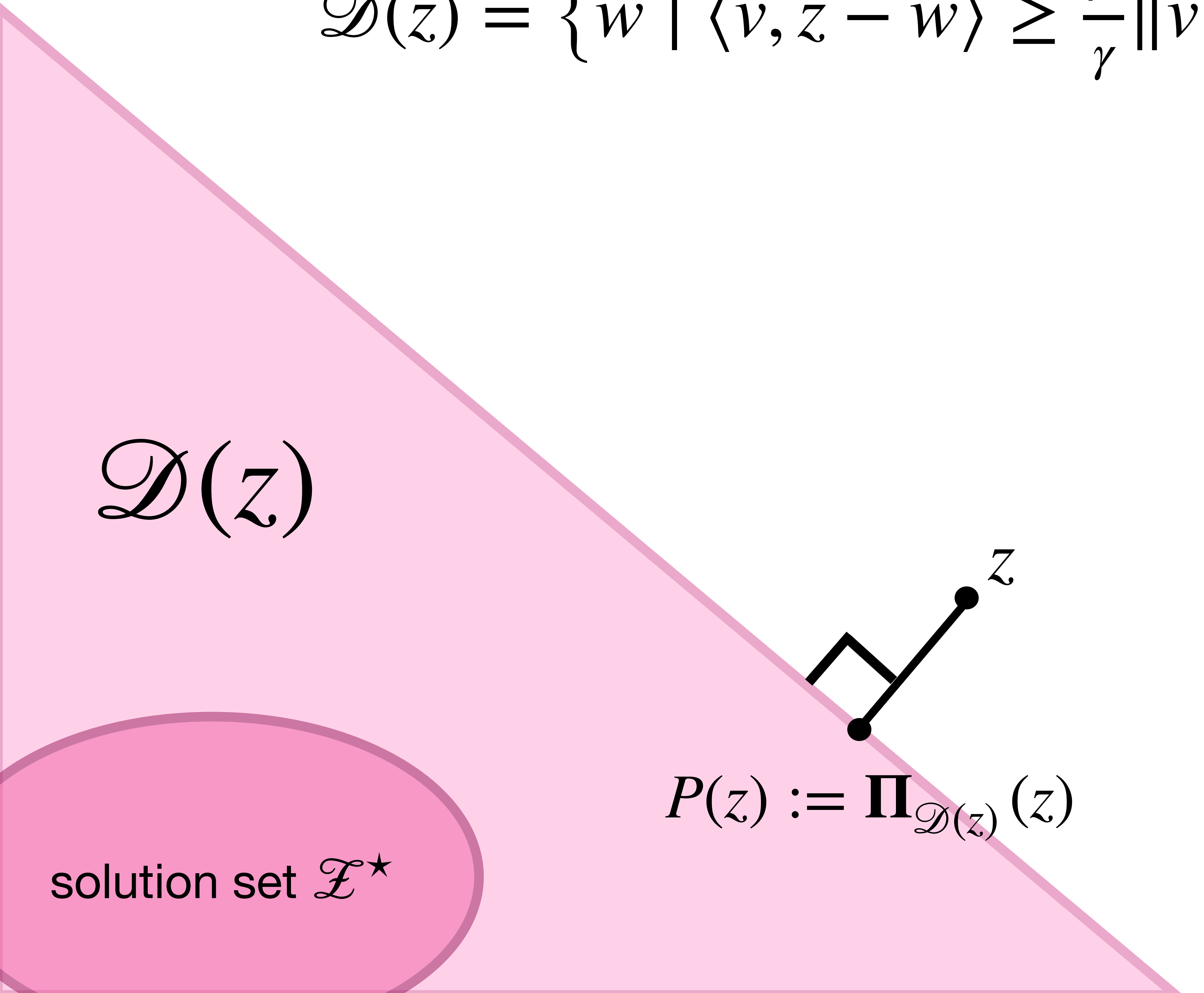
Inexact proximal point: Given $z \in \mathbb{R}^d$ find, for some error $\varepsilon \in \mathbb{R}^d$,

$$\bar{z} = z - (\bar{v} + \varepsilon) \quad \text{and} \quad \bar{v} \in \gamma S \bar{z}$$

Halfspace construction:

$$\mathscr{D}(z) = \left\{ w \mid \langle \bar{v}, \bar{z} - w \rangle \geq \frac{\rho}{\gamma} \|\bar{v}\|^2 \right\}$$

- Contains the solution set, i.e. $\mathscr{Z}^\star \subseteq \mathscr{D}(z)$
- The projection $P(z) := \mathbf{\Pi}_{\mathscr{D}(z)}(z)$ moves z towards the solutions
- We just need to convince ourselves that $\operatorname{fix} P \subseteq \operatorname{zer} S$!



Lemma Suppose the following relative error condition is satisfied

$$-\langle \varepsilon, \bar{v} \rangle \leq \sigma \|\bar{v}\|^2, \qquad \sigma \in [0, 1 + \tfrac{\rho}{\gamma})$$

Then

(i) P is firmly quasi-nonexpansive.

(ii) $P(z) = z - \alpha \bar{v}$ with $\alpha = \frac{\langle \bar{v}, z - \bar{z} \rangle + \rho/\gamma \|\bar{v}\|^2}{\|\bar{v}\|^2} \geq 1 + \frac{\rho}{\gamma} - \sigma$

(iii) $\mathscr{Z}^\star \subseteq \operatorname{fix} P \subseteq \operatorname{zer} S$

Dictates the range of ρ

A Hybrid method

Assumption (Weak MVI)

$\langle v, z - z^\star \rangle \geqslant \rho \|v\|^2$ for all $(v, z) \in \operatorname{grph} S$

Implicit method

$$
\begin{aligned}
\text{find} \quad & \bar{z}^k \in \mathbb{R}^d \quad \text{and} \quad \bar{v}^k \in \gamma S \bar{z}^k \\
\text{s.t.} \quad & \bar{z}^k = z^k - (\bar{v}^k + \varepsilon^k) \quad \text{and} \quad -\langle \varepsilon^k, \bar{v}^k \rangle \leq \sigma \|\bar{v}^k\|^2 \\
\text{update} \quad & z^{k+1} = z^k - \lambda_k \alpha_k \bar{v}^k \quad \alpha_k = \frac{\langle \bar{v}^k, z^k - \bar{z}^k \rangle + \delta/\gamma \|\bar{v}^k\|^2}{\|\bar{v}^k\|^2}
\end{aligned}
$$

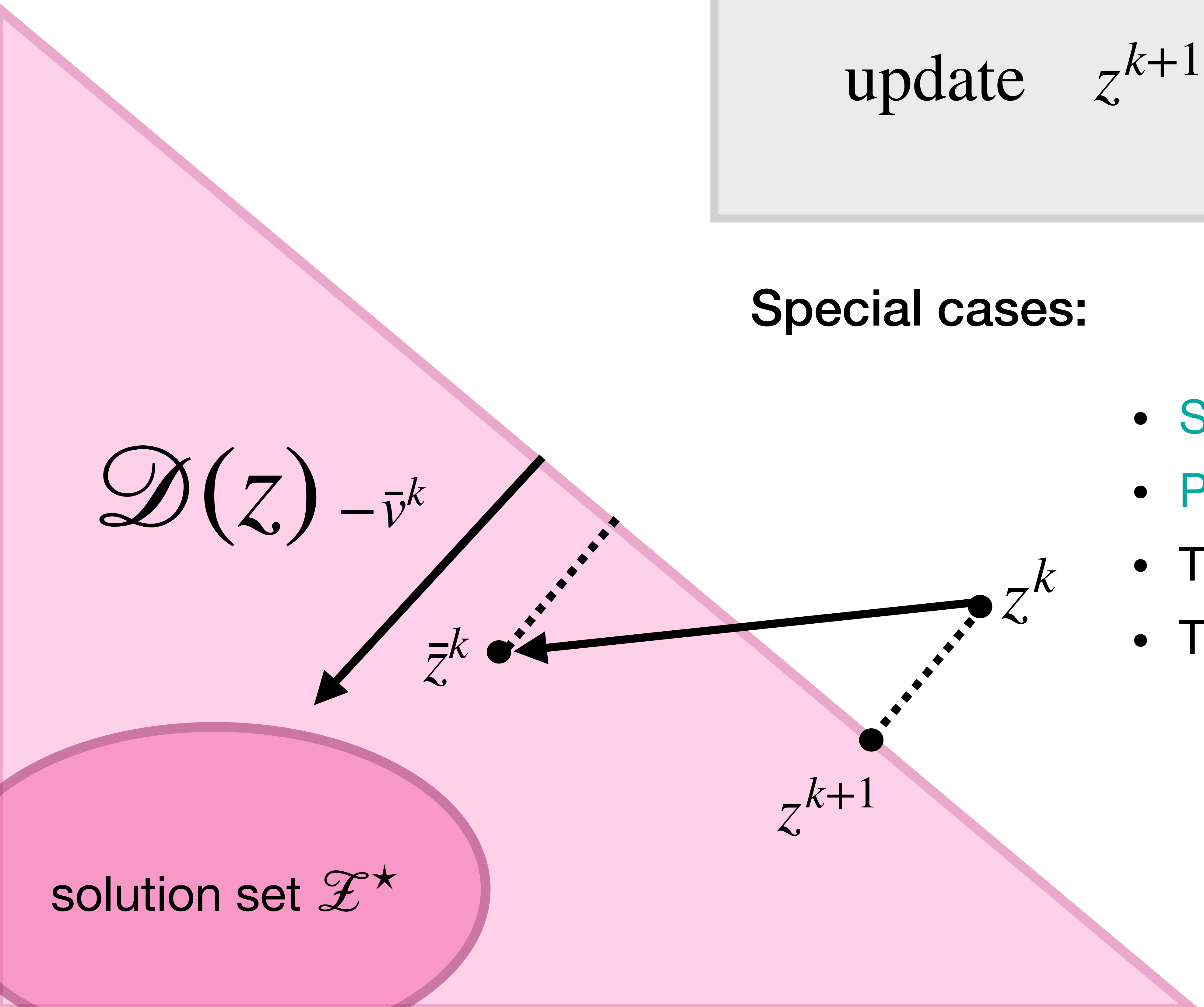
Parameters

$\sigma \in [0, 1+\frac{\delta}{\gamma})$

$\delta \leq \rho$

$\lambda_k \in (0,2)$

Special cases:



- Solodov and Svaiter [1999] ($\rho = 0$, apply Cauchy-Schwarz on error condition)
- P. et al. [2022] ($\varepsilon^k = \gamma(Fz^k - F\bar{z}^k)$)
- The relaxed proximal point algorithm ($\varepsilon^k = 0$)
- The relaxed extragradient method (by absorbing α_k into λ_k)

A Hybrid method

Assumption (Weak MVI)

$\langle v, z - z^\star \rangle \geqslant \rho \|v\|^2$ for all $(v, z) \in \operatorname{grph} S$

Implicit method

$$
\begin{array}{ll}
\text{find} & \bar{z}^k \in \mathbb{R}^d \quad \text{and} \quad \bar{v}^k \in \gamma S \bar{z}^k \\
\text{s.t.} & \bar{z}^k = z^k - (\bar{v}^k + \varepsilon^k) \quad \text{and} \quad -\langle \varepsilon^k, \bar{v}^k \rangle \leq \sigma \|\bar{v}^k\|^2 \\
\text{update} & z^{k+1} = z^k - \bar{\alpha} \bar{v}^k
\end{array}
$$

Nonadaptive stepsize

Theorem (informal) The scheme converges if $\rho > -(1 + \sigma + \bar{\alpha}/2)\gamma$.

Proof.

$$
\begin{aligned}
\|z^{k+1} - z^\star\|^2 &= \|z^k - z^\star\|^2 + \bar{\alpha}^2 \|\bar{v}^k\|^2 - 2\bar{\alpha}\langle \bar{v}^k, z^k - z^\star \rangle \\
&= \|z^k - z^\star\|^2 + \bar{\alpha}^2 \|\bar{v}^k\|^2 - 2\bar{\alpha}\langle \bar{v}^k, z^k - \bar{z}^k \rangle - 2\bar{\alpha}\langle \bar{v}^k, \bar{z}^k - z^\star \rangle \\
&= \|z^k - z^\star\|^2 - 2\bar{\alpha}(1 - \tfrac{\bar{\alpha}}{2}) \|\bar{v}^k\|^2 - 2\bar{\alpha}\langle \bar{v}^k, \varepsilon^k \rangle - 2\bar{\alpha}\langle \bar{v}^k, \bar{z}^k - z^\star \rangle \\
\text{(error condition)} &\leq \|z^k - z^\star\|^2 - 2\bar{\alpha}(1 - \sigma - \tfrac{\bar{\alpha}}{2}) \|\bar{v}^k\|^2 - 2\bar{\alpha}\langle \bar{v}^k, \bar{z}^k - z^\star \rangle \\
\text{(weak MVI)} &\leq \|z^k - z^\star\|^2 - 2\bar{\alpha}\underbrace{(1 - \sigma - \tfrac{\bar{\alpha}}{2} + \tfrac{\rho}{\gamma})}_{\text{Needs to be positive}} \|\bar{v}^k\|^2
\end{aligned}
$$

□

A Hybrid method (explicit)

Find $z \in \mathbb{R}^d$ such that $0 \in Fz + Az =: Sz$

Proximal point: Given $z \in \mathbb{R}^d$ find

$$z' = (\mathrm{id} + \gamma S)^{-1} z = (\mathrm{id} + \gamma A)^{-1}(z - \gamma F z')$$

Approximate with fixed point iteration

$$Q_z : \bar{z} \mapsto (\mathrm{id} + \gamma A)^{-1}(z - \gamma F \bar{z})$$

Algorithm 1 (Explicit method)

Repeat for $k = 0, 1, \ldots$
1. $\bar{z}^k \leftarrow z^k$
2. **repeat**
3. $\quad h^k \leftarrow z^k - \gamma F \bar{z}^k$
4. $\quad \bar{z}^k \leftarrow (\mathrm{id} + \gamma A)^{-1} h^k$
5. **until** $\langle z^k - \bar{z}^k, \bar{v}^k \rangle \geq (1-\sigma)\|\bar{v}^k\|^2$ where $\bar{v}^k = h^k - \bar{z}^k + \gamma F \bar{z}^k$
6. $z^{k+1} = z^k - \lambda_k \alpha_k \bar{v}^k$ with $\alpha_k = \dfrac{\langle \bar{v}^k, z^k - \bar{z}^k \rangle + \frac{\delta}{\gamma}\|\bar{v}^k\|^2}{\|\bar{v}^k\|^2}$

error condition

$\bar{v}^k \in \gamma S \bar{z}^k$

For $A \equiv 0$ when error condition passes **immediately** $\Rightarrow$

$$\bar{z}^k = z^k - \gamma F z^k$$
$$\bar{z}^k = z^k - \lambda_k \alpha_k \gamma F \bar{z}^k$$

(relaxed extragradient)

How quickly can the inner loop satisfy the ***error condition****?*

- When $\rho > -1/2L$ the error condition can pass **immediately**!
- More inner iteration leads to more relaxed condition on ρ through σ

Theorem (informal) Suppose Assumption 1-3 holds. Then

$$\min_{k \in \{0,\ldots,K-1\}} \mathrm{dist}(0, Sz^k)^2 \leq \frac{\|z^0 - z^\star\|^2}{\kappa\gamma^2(1 + \frac{\delta}{\gamma} - \sigma)^2 K}$$

The sufficient number of inner iteration n:

(i) for $\rho > -\frac{1}{2L}$ error condition can pass immediately ($n = 1$)

(ii) for $\rho > -\frac{1}{L}$ there exists a finite n for which the error condition passes

In contrast with the monotone case:
the extragradient approximation is not for free!
(the approximation error trades off the ρ range)

Oracle complexity

Key takeaway We can satisfy the error condition in a *finite* number of inner iterations

Corollary (informal)

The explicit scheme (Algorithm 1) achieves $\min_{k \in \{0,\ldots,K-1\}} \operatorname{dist}(0, S\bar{z}^k)^2 \leq \epsilon$ after at most

$$\#(\textbf{oracle calls}) \leq \frac{c\|z^0 - z^\star\|^2}{\gamma^2(1 + \frac{\delta}{\gamma} - \sigma)^2 \epsilon}$$

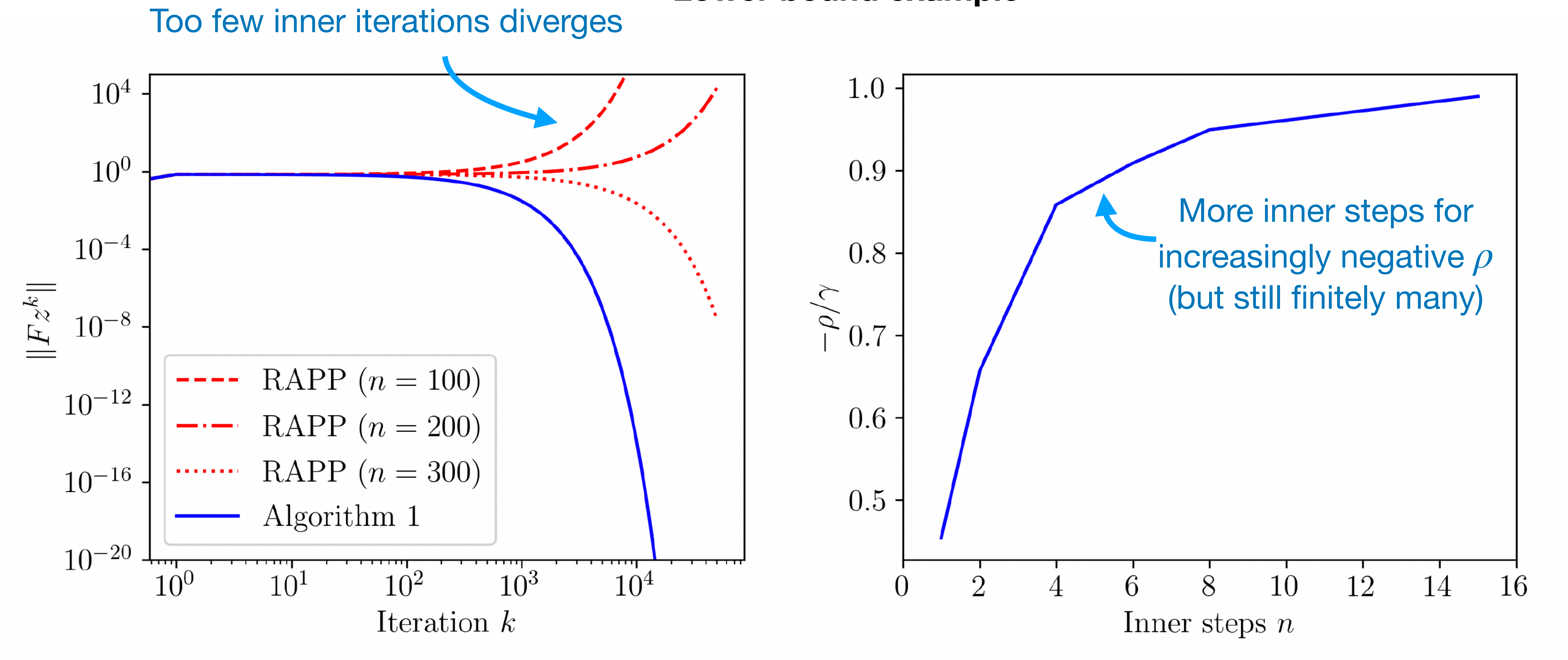
No log factor in ϵ !

to both the operator F and the resolvent $(\mathrm{id} + \gamma A)^{-1}$ where $c = \tilde{\mathcal{O}}(\frac{1}{1 + \rho L})$.

- Improves the complexity from $\mathcal{O}(\frac{1}{\epsilon} \ln \frac{1}{\epsilon})$ to $\mathcal{O}(\frac{1}{\epsilon})$
- Removes the need for prespecifying the number of inner steps n and the stepsize α_k.

Experiments

Lower bound example



Open problems

Last iterate

- Extragradient can converge (but only $\rho > -1/8L$)
  [Gorbunov et al., 2022]
- Relaxed inexact prox converges (but suffers log factor)
  [Alacaoglu et al., 2024, P. et al., 2023]

Can relaxed extra gradient enjoy last iterate guarantees (under cohypomonotonicity)?

Stochastic

- Increasing batch size
  [Diakonikolas et al., 2021, Alacaoglu et al., 2024]
- Lipschitz continuous in mean
  [P. et al., 2023]

Can increasing batch size be avoided without additional assumptions?

Single-call with constraints

- Unrelaxed method (so restricted ρ)
  [Cai & Zheng 2023]
- Relaxed method (but unconstrained)
  [Böhm, 2022]

Can a single-call method converge for $\rho > -1/2L$ with constraints?

Halpern

- Anchoring (but only for $\rho > -1/2L$)
  [Lee & Kim, 2021]
- Inexact Halpern (but suffers logarithmic)
  [Alacaoglu et al., 2024]

Can we shave off the logarithmic factor for inexact Halpern?